International Journal of Engineering Technology Science and Research
IJETSR
www.ijetsr.com
ISSN 2394 – 3386
Volume 4, Issue 7
July 2017

Some Results on Polar Derivative of a Polynomial

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ABSTRACT

In this paper, we present a result concerning to the polar derivative of a polynomials. Our theorem includes as special cases several interesting refinements and generalizations of some already known results.

Keywords: ***Polar derivative, Polynomial, Zeros .***

I.INTRODUCTION

Let P_n denote the class of all complex polynomials of degree at most n. If $P \in P_n$ then concerning the estimate of $|P'(z)|$ on $|z|=1$, we have

$$|P'(z)| \leq \max_{|z|=1} |P(z)| \qquad \ldots\ldots\ldots\ldots\ldots (1)$$

Inequality (1) is a famous result due to Bernstein [2], who proved it in 1912.
It is worth to mention that equality holds in (1) if and only if $P(z)$ has all its zeros at the origin, so it is natural to seek improvements under appropriate assumption on the zeros of $P(z)$. If we restrict ourselves to the class of polynomials $P(z)$ having no zeros in $|z|<1$, then (1) can be replaced by

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)| \qquad \ldots\ldots\ldots\ldots\ldots (2)$$

Where as if $P(z)$ has no zeros in $|z|>1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{2} \max_{|z|=1} |P(z)| \qquad \ldots\ldots\ldots\ldots\ldots (3)$$

Inequality (2) was conjectured by $Erdös$ and later verified by Lax [3], whereas inequality (3) is due to $Tura'n$ [5].

Aziz was among the first to extend some of the above inequalities by replacing the derivative with the polar derivatives of polynomials. In fact in 1988, Aziz [1], extended (2) to the polar derivative of a polynomial and proved that if $P \in P_n$, and $|P(z)| \neq 0$ in $|z|<1$, then for every complex number α with $|\alpha| \geq 1$,

$$\max_{|z|=1} |D_\alpha P(z)| \leq \frac{n}{2}\left(|\alpha|+1\right) \max_{|z|=1} |P(z)| \qquad \ldots\ldots\ldots\ldots\ldots (4)$$

Liman, Mohapatra and Shah [4], proved the following result in this direction:

Theorem A. Let $Q \in P_n$ and $Q(z)$ has all its zeros in $|z| \le 1$ and $P(z)$ be a polynomial of degree at most n If $|P(z)| \le |Q(z)|$ for $|z| = 1$ then for all $\alpha, \beta \in C$ with $|\alpha| \ge 1$, $|\beta| \le 1$,

$$\left| zD_\alpha P(z) + n\beta\left(\frac{|\alpha|-1}{2}\right)P(z) \right| \le \left| zD_\alpha Q(z) + n\beta\left(\frac{|\alpha|-1}{2}\right)Q(z) \right|, \quad for\ |z| \ge 1. \qquad \ldots\ldots\ldots\ldots\ldots (5)$$

II. MAIN RESULTS

In this paper, we shall prove the following result which as special cases give interesting refinement of (5). More precisely, we prove

Theorem 1. If $Q \in P_n$ and $Q(z)$ has all its zeros in $|z| \le 1$ and $P(z)$ be a polynomial of degree at most n If $|P(z)| \le |Q(z)|$ for $|z| = 1$ then for all $\alpha, \beta \in C$ with $|\alpha| \ge 1$, $|\beta| \le 1$, $m = \min_{|z|=1} |P(z)|$ *and* $|z| \ge 1$,

$$\left| zD_\alpha P(z) + n\beta\left(\frac{|\alpha|-1}{2}\right)P(z) \right| \le \left| zD_\alpha Q(z) + n\beta\left(\frac{|\alpha|-1}{2}\right)Q(z) \right| - mn\left\{ \left| \alpha + \beta\left(\frac{|\alpha|-1}{2}\right) \right| - \left| z + \beta\left(\frac{|\alpha|-1}{2}\right) \right| \right\}$$

Remark 1. If we put $m = 0$, in the above result, we get Theorem A.

III. LEMMAS

For the proof of the theorem, we shall make use of the following lemma.

By applying inequality (5) to the polynomials $P(z)$ and $z^n \min_{|z|=1} |P(z)|$, we get the following result.

Lemma 1. If $P \in P_n$ and $P(z)$ has all its zeros in $|z| \le 1$ then for all $\alpha, \beta \in C$ with $|\alpha| \ge 1$, $|\beta| \le 1$,

$$\left| zD_\alpha P(z) + n\beta\left(\frac{|\alpha|-1}{2}\right)P(z) \right| \ge n|z|^n \left| \alpha + \beta\left(\frac{|\alpha|-1}{2}\right) \right| \min_{|z|=1} |P(z)|, \quad for\ |z| \ge 1.$$

IV. PROOF OF THE THEOREM

Proof of Theorem 1. Let $m = \min_{|z|=1} |P(z)|$. If $P(z)$ has a zero on $|z| = 1$, then m=0 and the result follows by Theorem A. Henceforth, we suppose that all the zeros of $P(z)$ lie in $|z| > 1$ and so m>0. We have $|\lambda m| < |P(z)|$ on $|z| = 1$ for any λ with $|\lambda| < 1$. It follows by Rouche's theorem that the polynomial $G(z) = P(z) - \lambda m$ has no zeros in $|z| < 1$. Therefore the polynomial $H(z) = z^n \overline{G(1/\overline{z})} = Q(z) - \overline{\lambda} m z^n$ will have all its zeros in $|z| \le 1$. Also $|G(z)| = |H(z)|$ for $|z| = 1$. On applying Theorem A, we get for any $\alpha, \beta \in C$ with $|\alpha| \ge 1$, $|\beta| \le 1$,

$$\left|zD_\alpha G(z)+n\beta\left(\frac{|\alpha|-1}{2}\right)G(z)\right|\le\left|zD_\alpha H(z)+n\beta\left(\frac{|\alpha|-1}{2}\right)H(z)\right|,\quad for\ |z|\ge 1.$$

Equivalently

$$\left|zD_\alpha P(z)+n\beta\left(\frac{|\alpha|-1}{2}\right)P(z)-\lambda mn\left(z+\beta\left(\frac{|\alpha|-1}{2}\right)\right)\right| \le\left|zD_\alpha Q(z)+n\beta\left(\frac{|\alpha|-1}{2}\right)Q(z)-\bar{\lambda} mnz\left(\alpha+\beta\left(\frac{|\alpha|-1}{2}\right)\right)\right|. \quad \ldots\ldots\ldots\ldots\ldots (6)$$

Since $Q(z)$ has all zeros in $|z|\le 1$ and $\min_{|z|=1}|Q(z)|=\min_{|z|=1}|P(z)|$, by Lemma 1, we have for $|z|\ge 1$,

$$\left|zD_\alpha Q(z)+n\beta\left(\frac{|\alpha|-1}{2}\right)Q(z)\right|\ge n\left|\alpha+\beta\left(\frac{|\alpha|-1}{2}\right)\right|m|z|^n . \quad \ldots\ldots\ldots\ldots\ldots (7)$$

Now by choosing a suitable argument of λ in right hand side of (6), in view of (7), we get for $|z|\ge 1$,

$$\left|zD_\alpha P(z)+n\beta\left(\frac{|\alpha|-1}{2}\right)P(z)\right|-|\lambda|mn\left|z+\beta\left(\frac{|\alpha|-1}{2}\right)\right| \le\left|zD_\alpha Q(z)+n\beta\left(\frac{|\alpha|-1}{2}\right)Q(z)\right|-|\lambda|mn\left|\alpha+\beta\left(\frac{|\alpha|-1}{2}\right)\right|.$$

Letting $|\lambda|\to 1$, we get for $|z|\ge 1$,

$$\left|zD_\alpha P(z)+n\beta\left(\frac{|\alpha|-1}{2}\right)P(z)\right|-mn\left|z+\beta\left(\frac{|\alpha|-1}{2}\right)\right| \le\left|zD_\alpha Q(z)+n\beta\left(\frac{|\alpha|-1}{2}\right)Q(z)\right|-mn\left|\alpha+\beta\left(\frac{|\alpha|-1}{2}\right)\right|.$$

This implies

$$\left|zD_\alpha P(z)+n\beta\left(\frac{|\alpha|-1}{2}\right)P(z)\right| \le\left|zD_\alpha Q(z)+n\beta\left(\frac{|\alpha|-1}{2}\right)Q(z)\right|-mn\left\{\left|\alpha+\beta\left(\frac{|\alpha|-1}{2}\right)\right|-\left|z+\beta\left(\frac{|\alpha|-1}{2}\right)\right|\right\}.$$

Which completes the proof of Theorem1.

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