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Moments of Wavelet Packets Approximations of some Smooth Functions

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Abstract
We study the quadrature formulae for approximation of smooth functions using wavelet packets. Our approach to find these results is the generalization of the techniques which were developed by Sweldens and Piessions[16] using wavelets.
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1. INTRODUCTION

Definition A multiresolution analysis for $L^2(R)$ consists of a sequence of closed subspaces $\{V_j : j\in Z\}$ of $L^2(R)$ satisfying the following properties :

$V_j \subset V_{j-1}, \forall j \in Z$; ----------------------- 1.1

$\bigcap_{j\in Z} V_j = \{0\}$; ----------------------- 1.2

$\overline{\bigcup_{j\in Z} V_j} = L^2(R)$ ------------------------- 1.3

For every $f \in L^2(R), f(x) \in V_j$ <===> $f(2x) \in V_{j+1}, \forall j \in Z$ --------- 1.4

There exists a function $\emptyset \in V_0$ such that $\{\emptyset(x-k): k \in Z\}$ is orthonormal basis of V_0. 1.5

If we are given a multiresolution analysis, (1.4) implies a dilation equation

$$\phi(x) = \sqrt{2}\sum_{k\in Z} h_k\, \phi(2x-k)\ .\ .\ .\ .\ . \qquad (1.6)$$

Where $h_k = \left({}^1/_{\sqrt{2}}\right)\int_{-\infty}^{\infty} \emptyset(x)\overline{\emptyset(2x-k)}\, d$

Thus , the Fourier transform of 1.6 implies

$$\hat{\phi}(\xi) = m_0\left(\frac{\xi}{2}\right)\hat{\phi}\left(\frac{\xi}{2}\right).\ .\ .\ .\ .\ . \qquad (1.7)$$

Such that

$$m_0\left({}^{\xi}/_2\right) = \left({}^1/_{\sqrt{2}}\right)\sum_{k\in Z} h_k e^{(ik\ )}/_2$$

Where $\hat{\phi}(\xi) = \int_{-\infty}^{\infty} e^{i}\ \ \phi(x)\, d$

This gives $\hat{\phi}(\xi) = m_0\left(\frac{\xi}{2}\right)m_0\left(\frac{\xi}{2^2}\right)m_0\left(\frac{\xi}{2^3}\right)\dots\dots\dots\dots m_0\left(\frac{\xi}{2^j}\right)\hat{\phi}\left(\frac{\xi}{2^j}\right)$

If $j \to \infty$ and $\hat{\phi}(0) = 1$, then

$$\hat{\phi}(\xi) = \prod_{k=1}^{\infty} m_0(2^{-k}\xi)\, m_0\left(\frac{\xi}{2}\right)\hat{\phi}\left(\frac{\xi}{2}\right) \dots..\ .\ .\ .\ . \qquad (1.8)$$

The sequence h_n and its Fourier series $m_0(\xi)$ needs to satisfy specific constraints that are related to the properties of ϕ:

(i) If $\phi \in L^1$, then $\sum_n h_n = \sqrt{2}\, m_0(0) = \sqrt{2}$

(ii) ϕ is compactly supported in [p,q] if and only if h_n=0 for n<p or n>q

(iii) When the two previous constrains are satisfied, the orthogonality of $\{\phi(x-k): k \in Z\}$ is ensured if and only if

$$|m_0(\xi)|^2 + |m_0(\xi+\pi)|^2 = 1, \ldots \qquad 1.9$$

and there exists a compact set K, congruent to [$-\pi, \pi$] modulo 2 π, such that $m_0(2^{-k}\xi) \neq 0$ for all $\xi \in K$ and k>0.

We then define $m_1(\xi) = e^{i\cdot}\ \overline{m_0(\xi+\pi)}$ ……………………………………………………1.10

And $\hat{\psi}(\xi) = m_1\left(\frac{\xi}{2}\right)\hat{\phi}\left(\frac{\xi}{2}\right)$

where $\psi(x) = \sqrt{2}\sum_{k\in Z} g_k\, \phi(2x-k)$ …………………………………………..1.11

with $g_k = -1^{k+1}\overline{h_{1-k}}$.

Denoting by W_j the orthogonal complement of V_j in V_{j+1} , it is easy to show that $\{\phi(x-k): k \in Z\}$ is an orthogonal basis of W_0 . An obvious recalling shows that $\{\psi_{j,k} = 2^{j/2}\,\psi(2^j x - k): k \in Z\}$ is an orthonormal basis of W_j . Since $\bigcup_{j\in Z} V_j$ is dense in $L^2(R)$, the collection $\{\psi(j,k)\}_{j,k\in Z}$ is an orthonormal basis of $L^2(R)$

The decomposition of V_1 in to V_0 and W_0 as expressed in equation(1.6) and (1.11), reveals a more general "splitting trick" : if $\{e_k\}_{k\in Z}$ is an orthogonal basis of a Hilbert space H and if h_k and g_k are coefficients that satisfy previous constraints, then the sequences

$$u_k = \sqrt{2}\sum_n h_{n-2k}\, e_n$$

and

$$v_k = \sqrt{2}\sum_n g_{n-2k}\, e_n$$

are orthonormal basis of two orthogonal closed subspaces U and V such that H = U⨁V.

Wavelet packets are obtained by using the trick to split the W_j spaces. More precisely, one can define a family $\{\omega_n\}_{n\in Z}$ by taking $\omega_0 = \phi, \omega_1 = \psi$ and applying the following recursion

$$\left.\begin{aligned} \omega_{2n}(x) &= \sqrt{2}\sum_{k\in Z} h_k \omega_n(2x-k) \quad \text{and} \\ \omega_{2n+1}(x) &= \sqrt{2}\sum_{k\in Z} g_k \omega_n(2x-k) \end{aligned}\right\} \ldots\ldots (1.12)$$

The family $\{\omega_n(x-k)\}_{k\in Z} = \{\omega_{n,k}\}$ for $2^j \leq n \leq 2^{j+1}$ are the results of j-splitting of the spaces W_j more or less than j-times. For a family of wavelet packets $\{\omega_n\}$ corresponding to some orthogonal scaling function $\phi = \omega_0$, defines the family of subspaces of $L^2(R)$ given by

$$U_j^n = \overline{s\ \ \{\omega_n(2^j x - k): k \in Z\}}\ , \quad j \in Z,\ n = 0,1,2,\ldots \qquad 1.\ 13$$

Here we observe that

$$\left.\begin{array}{l} U_j^0 = V_j \\ U_j^1 = W_j \end{array}\right\} \quad \ldots\ldots \quad 1.14$$

So that the orthogonal decomposition $V_{j+1} = V_j \oplus W_j$ can be written as

$$U_{j+1}^0 = U_j^0 + U_j^1, j \in Z \quad \ldots\ldots \quad .1.15$$

This result can be generalized for other values of n=1,2,3,……………… as given below

$$U_{j+1}^n = U_j^{2n} + U_j^{2n+1}, j \in Z, \ldots\ldots \quad 1.16$$

Where U_j^n is defined by (1.13).

Since $V_j \to \{0\}$ a $j \to -\infty$, we se that

$$V_{j+1} = V_j \oplus W_j = \bigoplus_{j=-\infty}^{\infty} W_j, \ \forall\, j \in Z, \ldots\ldots \quad 1.17$$

Further, since $V_j \to L^2(R)$ as $j \to \infty$, we have

$$L^2(R) = \bigoplus_{j=-\infty}^{\infty} W_j, \ldots\ldots \quad 1.18$$

For each j=1,2,3, …………….. the decomposition trick (1.16) gives

$$\left.\begin{array}{c} W_j = U_j^1 = U_{j-1}^2 \oplus U_{j-1}^3 \\ W_j = U_j^1 = U_{j-2}^4 \oplus U_{j-2}^5 \oplus U_{j-2}^6 \oplus U_{j-2}^7 \\ \vdots \\ W_j = U_j^1 = U_{j-k}^{2^k} \oplus U_{j-k}^{2^k+1} \oplus \ldots \oplus U_{j-k}^{2^{k+1}-1} \\ \vdots \\ W_j = U_j^1 = U_0^{2^j} \oplus U_0^{2^j+1} \oplus \ldots \oplus U_0^{2^{j+1}-1} \end{array}\right\} \quad \ldots\ldots \quad .1.19$$

Where U_j^n , is defined in (1.13). Moreover, for each j = 1,2,3,…………….; k=1,2,3,……………

…,j and m = 0,1,…………, 2^k-1 , the set $\left\{2^{\frac{j-k}{2}}\omega_p\left(2^{j-k}x-1\right): j \in Z\right\}$ is an orthonormal basis of U_{j-k}^p where p=2^k+m. However, all the elements of these basis have the general form

$$\omega_j \ (x) = 2^{\frac{j}{2}}\omega_n\left(2^j x - k\right) \ldots\ldots \quad .1.20$$

If a function $f \in L^2(R)$, then

$$f \sim \sum_{n=2^p}^{2^{p+1}-1} \sum_{j,k \in Z} C_{l_1} \ \omega_{l_1} \ (x) \ . \ . \ . \ . \ . \ . \ . \ . \ (1.21)$$

where l=j-p, p = 0 if j <0 and p=0,1,2,. . . . .,j; if j≥ 0 ; will be a wavelet packet expansion of f and C_{l_1} the the wavelet packet coefficients defined as

$$C_{l_1} = \langle f, \omega_{l_1} \rangle \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad .1.22$$

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2. Wavelet Packets and their Moments

We define the moments of scaling function, wavelet and wavelet packet as

$$\left.\begin{array}{l} M_p = \int_{-\infty}^{\infty} x^p \, \varphi(x) d \\ N_p = \int_{-\infty}^{\infty} x^p \, \psi(x) d \quad , \\ R_p^n = \int_{-\infty}^{\infty} x^p \, \omega_n(x) d \end{array}\right\} \; p \geq 0, n \geq 0 \, . \quad . \quad . \quad . \qquad (2.1)$$

Theorem(2.1) Let M_p and R_p^n be the moments of scaling function and wavelet packet respectively as defined in (2.1) . Then

$$M_p = \sum_{n=0}^{2^l - i} \sum_{r=0}^{p} {}^pC_r \; k_0^r 2^{(p-1)l} G_{n,l} R_{p-r}^n$$

where ${}^pC_r$ are binomial coefficients .

$$G_{n,l} = G_{\varepsilon_1} G_{\varepsilon_2} \dots \dots \dots \dots . G_{\varepsilon_l}$$

with

$$n = \sum_{j=0}^{2^l - 1} \varepsilon_j 2^{j-1}, \varepsilon_j = \{0,1\}, \qquad G_\lambda(v) = \sum_l g_{-k+2l} v_k$$

Proof: we know that

$$M_p = \int_{-\infty}^{\infty} x^p \, \varphi(x) d$$

$$M_p = \int_{-\infty}^{\infty} x^p \, \omega_0(x) d$$

$$= < \omega_0(x), x^p >$$

Put $x = 2^j t - k$, then

$$M_p = < \omega_0\left(2^j t - k\right), (2^j t - k)^p >$$

$$\sum_{n=0}^{2^l - 1} \frac{1}{2^l} < \omega_0\left(2^j t - k\right), (2^j t - k)^p G_{n,l} >$$

for

$$n = \sum_{j=0}^{l} \varepsilon_j 2^{j-1}, \qquad \varepsilon_j = \{0,1\}, \qquad G_{n,l} = G_{\varepsilon_1} G_{\varepsilon_2} \dots \dots \dots \dots . G_{\varepsilon_l}$$

writing $\qquad 2^{j-1} t - k = x,$ we have

$$2^j t - k = 2(x + k) - k$$

$$(2^j x - k)^p = 2^p \; (x + k - k . 2^{-1})^p$$

$$= 2^p \; (x + k_0), \quad \text{where } \; k_0 = k(1 - 2^{-1})$$

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Hence

$$M_p = \sum_{n=0}^{2^l-1} \frac{1}{2^l} < \omega_n(x), 2^p \ (x+k_0)^p G_{n,l} >$$
$$= \sum_{n=0}^{2^l-1} \frac{1}{2^l} 2^p \ G_{n,l} \int_{-\infty}^{\infty} (x+k_0)^p \ \omega_n(x) d$$
$$= \sum_{n=0}^{2^l-1} 2^{(p-1)l} G_{n,l} \sum_{r=0}^{p} \int_{-\infty}^{\infty} {}^pC_r \ k_0^r x^{p-r} \omega_n(x) d$$
$$= \sum_{n=0}^{2^l-1} \sum_{r=0}^{p} {}^pC_r \ 2^{(p-1)l} \ k_0^r G_{n,l} R_{p-r}^n$$

Theorem(2.2): Let r be a non- negative integer . Let $\omega_n$be a wavelet packet in $C^r(R)$ such that

$$|\omega_n(x)| \le \frac{C}{(1+|x|)^{r+1+\varepsilon}}, \qquad f \quad sc \qquad \varepsilon > 0, \ \ . \ . \ . \ . \ . (2.2)$$

and $\omega_n^{(m)} \in L^{\infty}(R)$, (m-th derivative of ω_n) for m=1,2,3,………..,r.

If $\left\{2^{j/2}\omega_n(2^j x - k) : j, k \in Z\right\}$ is an orthonormal system in $L^2(R)$, then all moments of ω_n up to order $r$are zero for all n=1,2,3,…….. i.e.

$$\int_R x^m \omega_n(x) d \ \ = 0$$

Proof: We shall prove the theorem by induction. Let us first assume that r=0. Let 'a' be the dyadic number, $a = 2^{-j_0} k_0$ for some fixed $j_0, k_0 \in Z$ such that $\omega_n(a) \neq 0$,since $\|\omega_n\|_2 = 1$ and ω_n is continuous, such that 'a' exists. From Lemma[ ] we have

$$\int \overline{\omega_n(x)} \ \omega_n(2^j x - k) d \ \ = 0, j, k \neq 0$$

taking $k = 2^{j-j_0} k_0$ with $j > \max\{j_0, 0\}$. Then the equality becomes

$$\int_R \overline{\omega_n(x)} \ \omega_n\left(2^j (x-a)\right) d \ \ = 0$$

Let $= 2^j (x-a)$. Then

$$\int_R \overline{\omega_n(a + 2^{-j} y)} \ \omega_n(y) d \ \ = 0$$

Since , the left hand side tends to $\overline{\omega_n(x)} \int_R \omega_n(y) d$ when $j \to \infty$ and $\omega_n(a) \neq 0$, we have

$$\int_R \omega_n(y) d \ \ = 0$$

Before considering the integral case, let us prove the theorem when r=1 in order to explain the method in a simple context. Since

$$\int_R \omega_n(y) d \ \ = 0$$

$$\Gamma(x) = \int_{-\infty}^{x} \omega_n(y) d \quad \to 0 \ a \quad x \to \infty$$

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Moreover,

$$\Gamma(x) = \int_{-\infty}^{x} \omega_n(y)d \quad = -\int_{x}^{\infty} \omega_n(y)d$$

Since we assume that

$$|\omega_n(x)| \leq \frac{C}{(1+|x|)^{2+\varepsilon}}$$

for an $\varepsilon > 0$, then

$$\Gamma(x) \leq \frac{C_1}{(1+|x|)^{1+\varepsilon}}$$

Integrating by parts, we get

$$\int_{-\infty}^{\infty} \Gamma(x)\, d \quad = \int_{-\infty}^{\infty} x \quad _n(x)d$$

Hence it is sufficient to show that

$$\int_{-\infty}^{\infty} \Gamma(x)\, d \quad = 0$$

We can use the argument, we just used for $\omega_n(x)$. Since ω_n is not constant and ω'_n , derivative of ω_n, is continuous, there exists an $a = a^{-j_0}k_0$ so that $\omega'_n(a) \neq 0$. Let us write again

$$\int_R \overline{\omega_n(x)}\, \omega_n(2^j x - k)d \quad = 0$$

or

$$\int_R \overline{\omega_n(x)}\, \omega_n\left(2^j(x-a)\right)d \quad = 0$$

Then, on integration by parts, it follows that

$$\int_R \overline{\omega_n(x)}\, \Gamma\left(2^j(x-a)\right)d \quad = 0$$

which by a change of variables gives us

$$\int_R \overline{\omega'_n(a+2^{-j}x)}\, \Gamma(x)d \quad = 0$$

We conclude, letting $j \to \infty$, that

$$\int_{-\infty}^{\infty} x \quad _n(x)d \quad = -\int_{-\infty}^{\infty} \Gamma(x)\, d \quad = 0$$

Let us now proceed by induction from r-1 to r. Since all moments up to order r-1 are zero. We can integrate ω_n r-times and obtain functions $\Gamma'_1 = \Gamma, \Gamma_2, \ldots\ldots \Gamma_r$ such that $\Gamma'_1 = \Gamma_{1+1}$ and $\Gamma_1, \Gamma_2, \ldots\ldots \Gamma_r$ all tends to zero at $\pm\infty$. Moreover, there exists constant C_1 such that

$$|\Gamma_1(x)| \leq \frac{C_1}{(1+|x|)^{r-l+1+\varepsilon}}, l = 1,2, \ldots, r$$

Integrating by parts r-times, it is easy to see that

$$\int_R x^r \omega_n(x)d \quad = 0$$

is equivalent to

$$\int_R \Gamma_r(x)d \quad = 0$$

Again, since ω_n is not a polynomial and $\omega_n^{(r)}$ is continuous, there exists an $a = 2^{-j_0}k_0$ with $\omega_n^{(r)}(a) \neq 0$. From

$$\int_R \omega_n(x)\,\omega_n\left(2^j(x-a)\right) d \quad = 0$$

we obtain

$$\int_R \omega^r{}_n(x)\,\Gamma_r\left(2^j(x-a)\right) d \quad = 0$$

By integrating r-times. The desired result is obtained by changing the variables and letting $j \to \infty$

Theorem: (2.3) Let $\omega_n \in L^\omega(R)$, be a wavelet packet such that

$$|\omega_n(x)| \leq \frac{C}{(1+|x|)^{1+\varepsilon}}, \quad a.e.\ f \quad s \quad \varepsilon > 0$$

If $\{\omega_{j,n,k}: j,k \in Z, n \in Z^+\}$ is an orthonormal system in $L^2(R)$, then

$$\int_R \omega_n(x) d \quad = 0$$

Proof : Let a be Lebesgue point for ω_n such that $\omega_n(a) \neq 0$. Let a_j be a sequence of dyadic points $2^{-j}k$ such that $|a_j - a| < 2^{l-j}$. Then as before in theorem (2.2), we have

$$\int_R \overline{\omega_n(a_j + 2^{-j}x)}\,\omega_n(x) d \quad = 0$$

$$|a_j - a| < 2^{-j} \leq 2^{l-j} \ f \quad l \geq 0$$

$$u(l,j) \leq 2^{-l+j}\int_{a-2^{l-j+1}}^{a+2^{l-j+1}} |\omega_n(y) - \omega_n(a)| d \ ,$$

where a is a Lebesgue point, for every l, the sequence $\{u(l,j)\}_{j=1}^{\omega}$ tends to 0 as j tends to ∞.

Therefore

$$I_j = C\sum_{l \geq 0} 2^{-\varepsilon_l}\ u(l,j) \quad \to \ 0\ whe \quad j \to \infty$$

This completes the proof of the theorem.

Remark (2.4) Since

$$\widehat{\omega}_n(\xi) = \int_R e^{-i} \ \omega_n(x) d$$

$$\frac{d}{d}\{\widehat{\omega}_n(\xi)\} = \int_R \frac{d}{d}\{e^{-i} \ \}\omega_n(x) d$$

$$= \int_R -i \ e^{-i} \ \omega_n(x) d$$

$$\frac{d^2}{d^{\ 2}}\{\widehat{\omega}_n(\xi)\} = \int_R (-i \ )^2\, e^{-i} \ \omega_n(x) d$$

Remark (2.5). Above theorem shows that a compactly supported wavelet packet can be continuously differentiable up to some finite order n, where n is a non-negative integer.

Remark(2.6). From theorem (2.3) we can conclude that if ω_n is a wavelet packet with compact support. We should have

$$\int_{-\infty}^{\infty} \omega_n(x)d \;\; = 0$$

or

$$\widehat{\omega}_n(0) = 0$$

and similarly if ω_n is a compactly supported orthonormal wavelet packet such that $\omega_n \in C^m(R)$, we must have

$$\int_{-\infty}^{\infty} x^l \omega_n(x)d \;\; = 0, \qquad f \quad l = 0,1,2, \; . \; . \; . \; . \; . \; . \; , m$$

3…Quadrature Formulae

The general idea of quadrature formulae is to find weights t_k and abscissae x_k such that

$$\int_{-\infty}^{\infty} f(x)\omega_n(x)d \;\; = Q[f(x)] = \sum_{k=0}^{r} t_k x_k$$

Definition(3.1). The degree of accuracy of a quadrature formula is q if it yields the exact result for every polynomial of degree less than or equal to q.

The order of convergence is determined by the degree of accuracy as follows. If f(x) belongs to C^{q+1}, then

$$\frac{v_{j,k}^n - 2^{-n/2} Q[f(2^{-n}x)]}{v_{j,k}^n} = O[h^{q+1}],$$

where$h = 2^{-n}$, $v_{j,k}^n = < f, \widetilde{\omega}_{j,n,k} >$.This can easily be seen by Taylor's expansion.

Theorem(3.2). Let $\omega_n(x)$ be an orthonormal wavelet packet with N>1. Then

$$R_0^n R_2^n = (R_1^n)^2$$

Proof. Define

$$K_m = < x, \omega_n(x)\omega_n(x-m) >, \quad m \in Z$$

Because of the orthogonality the following holds

$$K_{-m} = < x - m, \omega_n(x-m)\omega_n(x) > = K_m$$

Consequently

$$0 = \sum_m m K_m = < x, \omega_n(x) \sum_m m\, \omega_n(x-m) > \; . \; . \; . \; . \; . (3.1)$$

and since N>1

$$\sum_m m\omega_n(x-m) = \sum_m (x-m)\omega_n$$
$$= x \sum_m \omega_n(m) - \sum_m m\omega_n(m)$$
$$= xR_0^n - R_1^n$$

Thus, from(3.1), we have

$$0 = \sum mK_m$$
$$= \; < x, \omega_n(x)\{xR_0^n - R_1^n\} >$$
$$= \sum x^2\, \omega_n(x) R_0^n - \sum_x x\, \omega_n(x) R_1^n$$

$$= R_0^n R_2^n - (R_1^n)^2$$

Hence

$$R_0^n R_2^n = (R_1^n)^2$$

Practical Aspect(3.3). In applications such as signal and image processing, discrete samples a_k usually are given. Then, there are several ways to start the multiresolution analysis. First, we can construct a function $a(x) \in U_j^n$

$$a(x) = \sqrt{h} \sum_k a_k \omega_{j,n,k}(x) \quad w \quad h\, h = 2^{-j}$$

We can see that the continuous function *a*(x) will in a way "follow " the discrete sample a_k. The quadrature formula can help us to find a relationship between the function *a*(x) and the discrete samples a_1. Indeed , using the biorthogonal notation

$$\sqrt{h} a_k = < a, \tilde{\omega}_{j,n,k} >$$

and

$$< a, \omega_{j,n,k} > = \sqrt{h}\left[a\left(h(R_1^n + k)\right) + O(h^t)\right],$$

so

$$a_k = a\left(h(R_1^n + k)\right) + O(h^t)$$

Hence *a*(x) satisfies the quasi-interpolating property.

Second we can consider the sample a_k as function evaluations $a_k = f(hk)$. This corresponds to the one point formula with $x_1 = 0$.

Theorem(3.4). Let $f(x) \in C^N$ with $f^i(x)$ be bound for i≤N. Then for $h = 2^{-j}$

$$\sum_k f(hk)\omega_n(2^j x - k) = \sum_k \omega_n(k) f(x - hk) + O(h^N).$$

Proof. We have

$$\sum_k f(hk)\omega_n(2^j x - k) = \sum_k f(x - (x - hk))\omega_n(2^j x - k)$$

$$= \sum_k \left[\sum_{i=0}^{N-1} \frac{\{-(x - hk)\}^i}{i!} f^i(x)\right] \omega_n(2^j x - k) + O(h^N)$$

$$= \sum_k \left[\sum_{i=0}^{N-1} \frac{\left\{(-h)^i (2^j x - k)^i\right\}}{i!} f^i(x)\right] \omega_n(2^j x - k) + O(h^N)$$

$$= \sum_{i=0}^{N-1} \frac{(-h)^i f^i(x)}{i!} \sum_k (2^j x - k)^i \omega_n(2^j x - k) + O(h^N)$$

$$= \sum_{i=0}^{N-1} \frac{(-h)^i f^i(x)}{i!} \sum_k k^i \omega_n(k) + O(h^N)$$

$$= \sum_k \left[\sum_{i=0}^{N-1} \frac{(-hk)^i f^i(x)}{i!}\right] \omega_n(k) + O(h^N)$$

$$= \sum_k f(x - hk)\, \omega_n(k) + O(h^N)$$

Thus theorem states that taking function evaluations as coefficients results in approximations a different function

$$\tilde{f}_j(x) = \sum_k f(x - hk)\omega_n(k) + O(h^N)$$

with an error of $O(h^N)$.

Construction Scheme(3.5). Here we assume that $\omega_n(x)$ has compact support [0,L] and satisfies a refinement equation (2.2.12) with L+1 non-zero coefficients h_k and g_k. Although the construction is general, we focus on wavelet packet function with compact support. Since in this case we have the extra limitation that the abscissae should fall inside the integration interval. We construct the r-point quadrature formula with $x_k = d_k - \tau$, $d_k = (k-1)2^S$ and $(r-1)2^S - L \le \tau \le 0$. The range of the shift τ is determined by the requirement that the abscissae should not fall outside the integration interval. To have a non-zero range for the shift τ the parameters r and s should be chosen such that $(r-1)2^S < L$.

Since there are L+1 unknowns $\{\tau, t_1^n, t_2^n, \ldots, t_r^n\}$, we can try to achieve a degree of accuracy r. This results in following system, which is non-linear in the unknown τ :

$$\sum_{k=1}^{r} t_k^n [d_k - \tau]^l = R_i^n, \qquad 0 \le i \le r \ldots\ldots (3.2)$$

The value of the shift τ can be determined polynomial $\prod(x)$. This polynomial is defined as

$$\prod(x) = \prod_{k=1}^{r}(x - x_k)$$

$$= \prod_{k=1}^{r}(x + \tau - d_k)$$

$$\sum_{i=0}^{r} p_i(\tau)x^i,$$

where $p_i(\tau)$ is a polynomial of degree r-1. Since the degree of accuracy is r, the quadrature formula gives the exact result for the product polynomial $\prod(x)$. So

$$0 = Q_r[\prod(x)] = \sum_{i=0}^{r} p_i(\tau)R_i^n = \Gamma(\tau)$$

The latter expansion is a polynomial of degree r in τ . For the quadrature formula to exist ,$\Gamma(\tau)$ must have a root in the interval $[(r-1)2^S - L, 0]$. However, the existence of such a root is not theoretically guaranteed. If there is not root in this interval, an arbitrary values of τ must be chosen and one degree of accuracy is lost. Once τ is determined, the weights are the solution of the linear system formed by r-equations of (3.2) . To construct $\Gamma(\tau)$ we write

$$P_i(\tau) = \sum_{j=0}^{r-i} p_{i,j}\tau^j$$

and

$$\Gamma(\tau) = \sum_{j=0}^{r}\left[\sum_{j=0}^{r-i} R_i^n p_{i,j}\right]\tau^j$$

The coefficients $p_{i,j}$ are symmetric $p_{i,j} = p_{j,i}$ since the product polynomial is symmetric in τ and x can be found as $P_{i,j}^{(r)}$ where

$$\prod^{(m)}(x) = \prod_{k=1}^{n}(x + \tau - d_k)$$

$$\sum_{i=0}^{m}\sum_{j=o}^{m-i} p_{i,j}^{(m)}\tau^j x^i$$

An algorithm to calculate $p_{i,j}$ can be derived by writing

$$\prod^{(m)}(x) = (x + \tau - d_m)\prod^{(m-1)}(x)$$

and identifying the coefficients of the powers of x and r. the dis advantage of this method is that the system of equations (3.2) is ill conditioned if r is large.

Modified Construction (3.6). The ill conditioning problem can be overcome if we use the basis of Chebyshev polynomial. This technique is also used successfully in [14,15]. The Chebyshev polynomial $T_n(x)$ of degree n is defined by $T_0(x) = 1, T_1(x) = x$ and $T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$ for all n≥2. Since the interesting properties of these polynomials only hold in the interval [-1,1], we transform the wavelet packet $\omega_n(x)$to this interval yielding a function $\omega_n^{□}(y)$. We will use the notation y to indicate an independent variable that varies between -1 and 1.

$$2\omega_n^{□}(y) = L\omega_n(x), \quad x = L\frac{(y+1)}{2}$$

The refinement equation becomes

$$\omega_{2n}^{□}(y) = \sqrt{2}\sum h_k\omega_n^{□}\left(2y - \frac{2k}{L} + 1\right),$$

$$\omega_{2n+1}^{□}(y) = \sqrt{2}\sum g_k\omega_n^{□}\left(2y - \frac{2k}{L} + 1\right),$$

we construct the quadrature formula

$$\int_{-1}^{1}\omega_n^{□}(y)\, f\left(\frac{L(y+1)}{2}\right)d \;\; = \int_{-1}^{1}\omega_n^{□}(y)f^{□}(y)d$$

$$= \sum_{k=1}^{r} t_k^n f^{□}(y_k)d$$

$$= Q_r[f(x)],$$

with

$$y_k = d_k^{□} - \tau^{□}, \qquad d_k^{□} = \frac{2d_k}{L-1} \;\; a \qquad \tau^{□} = \frac{2\tau}{L}.$$

Let $(R_p^n)^{□}$ denote the modified moments

$$(R_p^n)^{□} = \int_0^1 \Gamma_p(x)\omega_n^{□}(x)d$$

The new system can be written as

$$\sum_{k=1}^{r} t_k^n\Gamma_1(d_k^{□} - \tau^{□}) = (R_1^n)^{□}, \qquad 0 \le i \le r.$$

The solution procedure is similar to the one in the previous section. We construct a polynomial $\Gamma^{□}(\tau^{□})$, written as a linear combination of Chebyshevpolynomials, and to find a root in the approximate interval as in [16].

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