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Capacitated Santa Claus Problem

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ABSTRACT

Consider the problem in which we have k players and a set of n items, where, each player has a happiness value for every item, the problem is to distribute all the items among the players so as to maximize the fairness. This is known as the Santa Claus problem. There are several variants to this problem which are hard to analyze. The best known result for Uniform Case of Santa Claus problem is a 3/4 approximation and the best known result for Restricted Case of Santa Claus problem is 1/13 approximation. The driving motivation for our work was to analyze this large gap in the two cases and study new sets of problems that lie in the spectrum. Our main results include an x approximation for the new set of problems identified on the spectrum. These problems are the cases where every item can be liked by x number of players and each player has a limited capacity. In our work we also come up with OPT − vz approximation for the proposed algorithm for the general Santa Claus problem with additional capacity constraint.

Keywords: Constrained Allocation, Santa Claus, Capacitated Santa Claus problem

INTRODUCTION

The fair allocation of indivisible goods is a major problem in any system containing multiple resources, where each user in the system may have different preferences for each resource. One of the most popular allocation policies proposed so far has been Max-Min fairness, which maximizes the minimum allocation received by a user in the system. The fair allocation of indivisible goods among the persons is an NP complete problem. Each good has different preference value with respect to the persons who likes it. This problem of allocation of indivisible goods can be related to many more real world applications such as inheritance issues, divorce settlements, distributing uneven gifts to children, goods sharing and many others. In resource allocation to internet services in cellular network, it is expected that a huge amount of frequency or bandwidth will be allocated for internet applications in the system. Normally every person should get guaranteed minimum resources with fairness and maximum utilization of resources. Allocation of water resource is an another real issue in which all water users should be identified, and registered along with their agreed share of water allocated for abstraction or storage .In the context of resource allocation, fairness is a state in which each stakeholder's welfare is increased to the fullest extent possible subject to limited resources by properly accounting for different claims and individual circumstances. In the indivisible resource area, most of the issues are based on the Santa Claus problem, where interpreting the persons as kids and the resources as gifts leads to Santa's annual allocation problem of making the least happy kid as happy as possible. We mainly focusing on fair allocation of indivisible resources based on Santa Claus problem.

OVERVIEW AND PROBLEM STATEMENT

Consider a problem in which you are given many non-identical gifts and you are required to give them to some children who will cry if they feel they have received invaluable gifts. Under this situation how will you give the gifts to the children making sure that none of them starts crying? This problem is known as the fair allocation problem. Now consider another situation, where there are certain taxi drivers and customers, and you wish to make sure that maximum number of customers should get picked up by all the taxi drivers without the drivers having to travel more than a certain threshold.

Input: n kids D = {d1, d2... d_n}, m gifts C = {c1, c2... cm}, pair wise valuations v_{ij} for every kid-gift (i − j) pair Output: Allocation η: D → 2^C of gifts to kids Optimization: Maximize Φ, where Φ = fairness is depend on the likeness of kids to the gifts.

The problem states that given a set of n kids D = {d1, d2... dn} and a set of m gifts C = {c1, c2... cm}. The problem that we will be dealing in this paper is to allocate gifts to kids so as to maximize the fairness which makes all kids equally happy.

MOTIVATION

The problem of Constrained Fair Allocation has various aspects. It can be used in ridesharing domain, i.e. we can allocate customers to drivers or vice versa. If we think about the scope of the general allocation problem, not restricting to transporting people and think about using this concept to share goods which need to be transported, at this point we bring into picture not only cars but also various tempo, truck and bus agencies who make daily trips to several places. The divisibility of goods to be allocated plays a vital role. In case the goods are divisible then the problem is known as the cake cutting problem which is well studied in the literature. In the problem of allocating customers to drivers in a Ride Sharing scenario we obviously cannot divide a particular customer. So this problem becomes the problem of allocating indivisible goods. There has been lot of study regarding allocation of indivisible goods. This problem of allocation of indivisible goods can be related to many more real world applications other than Ride Sharing such as inheritance issues, divorce settlements, distributing uneven gifts to children, goods sharing and many others. More applications about allocation problem can be found over here [Brams and Taylor. 1996]. The aim of this project is to study, develop and design algorithms for constrained fair allocation of indivisible goods. This can be used in many real life applications where society gets benefited. The one of the main application of this algorithm is in the ride sharing domain. The benefits of ride sharing are substantial. A successful ride sharing scheme could, from a social perspective, reduce fuel consumption and emissions, reduce parking problems, reduce traffic problems, reduce journey time due to less traffic and reveal many other benefits. From a commuter's perspective it could reduce cost of the travel, reduce parking cost, reduce stress due to journey and reduce journey time. Most of the research that has been done on rides sharing revealed extreme benefits of rides sharing. For example a research [Hardesty, 2014] from National Academies of Sciences, researchers at MIT, Cornell University, and the Italian National Research Council's Institute for Informatics and Telematics 4 present a new technique that enabled them to exhaustively analyze 150 million trip records collected from more than 13,000 New York City cabs over the course of a year. Their conclusions were: If passengers had been willing to tolerate no more than five minutes in delays per trip, almost 95 percent of the trips could have been shared. The optimal combination of trips would have reduced total travel time by 40 percent, with corresponding reductions in operational costs and carbon dioxide emissions. The fair allocation of indivisible goods is an NP-Complete problem. We want to contribute to the field of algorithm by studying this NP-Complete problem and come up with better techniques to tackle the hardness of the NP Completeness. In our work we identify and study the Santa Claus problem and tried to come up with an approximation for Santa Claus problem with additional capacity constraint.

CAPACITATED SANTA CLAUS PROBLEM

Some of the real world applications of Santa Claus problem such as ride sharing; additional capacity constraint plays a vital role during the allocation. In this case passengers are allocated to drivers in such a way that to maximize the profit of the unlucky driver. Here the capacity is the seating capacity of the vehicle. So we can formulate the classic Santa Claus problem with additional capacity constraint to tackle these type real world applications. Here we are explaining the proposed problem and the approaches to solving it.

PROPOSED PROBLEM STATEMENT

Given n gifts g1, g2, g3... gn to be allocated to m kids k1, k2, k3... km. Here we use two vectors $S \in Z_n$, representing the size of the gifts and $C \in Z_m$ representing the capacities of the kids. Another vector $P \in Z$ mxn representing the profits earned by the kids by getting different gifts. The objective of this problem is to allocate the gifts among the kids in such a way that the minimum profit earned by an unlucky kid should be maximized.

AN APPROXIMATION ALGORITHM FOR CAPACITATED SANTA CLAUS PROBLEM USING LINEAR PROGRAMME

Since the classic Santa Claus problem itself is an NP-complete problem, the problem with additional capacity constraint is also NP-complete. Here we are suggesting an approach for developing an approximation algorithm for the proposed capacitated Santa Claus problem. The following part of the chapter contains the details of different steps involved in this approach.

LP FORMULATION OF THE PROBLEM

First formulate an integer program for this problem. An assignment ILP (P, S, C) may be created for this where $1 \leq i \leq m$ and $1 \leq j \leq n$. A $0-1$ assignment variable x_{ij} is also used. If any kid k_i get a gift g_j , then the value of x_{ij} will be 1 .The constraints on the program are that each item must be allocated exactly once and that each players utility must be at least as much as the objective function within the specified capacity of each player. Thus the optimal allocation for the capacitated Santa Claus problem instance is the solution to the integer program: Hence the ILP (P, S, C) can be represented as follows: Let, x_{ij} be a variable which has value 1 if kid i gets gift j otherwise 0. So we can formulate the integer program as,

Max.T such that. $\forall j: \sum x_{ij} = 1$, $\forall i: T \leq \sum Pij.xij$, $\forall i: C_i \geq \sum S_j .x_{ij}$, $\forall i \forall j: x_{ij} \in \{0, 1\}$

The constraints in the integer program ensure that each gift is assigned to at most one kid and every kid should get a minimum value T without violating their own capacity constraint. Solving the integer program is an NP-complete problem. So we are using the linear programming relaxation of the given integer program which will result in a fractional allocation. The corresponding LP relaxation which we denote by LPR (P, S, C) is given by Max.T s.t. $\forall j: \sum xij = 1$, $\forall i : T \leq \sum Pij.xij$, $\forall i : Ci \geq \sum Sj .xij$, $\forall i, \forall j : x_{ij} \geq 0$ Now we may obtain a feasible solution of LP(P, S, C) by rounding the optimal solution of LPR(P, S, C) which can be found out in polynomial time. Let X^* be an optimal extreme point solution of LPR (P, S, C). Since it is a basic feasible solution it must contain at most $l \leq 2m + n$ non zero basic components and the remaining $mn - l$ variables are set to zero; which corresponds to the non basic components of LPR(P, S, C) Lenstra et al.

CONSTRUCT A BIPARTITE GRAPH

Create a bipartite graph G: (A, B, E) where A = {1, 2, 3... m} representing the set of kids B = {1, 2, 3, ..., n} representing the set of gifts $E = \{(i, j)|x_{ij} \geq 0, 1 \leq i \leq m, 1 \leq j \leq n\}$ .The edge set contain edge is labeled by a value corresponding to its Xij. Clearly G contains exactly l edges with non zero values. We are using the rounding method illustrated in Lenstra et al. for rounding the fractional LP solution to the integer solution. In that case they considered the Santa Claus problem without capacity constraint. So that there is only at most m + n edges present in the bipartite graph. For using the same rounding here we have eliminate enough edges until we get the bipartite graph with at most m + n edges.

ELIMINATING SOME EDGES FROM THE GRAPH

For removing the extra edges from the bipartite graph created using the fractional LP solution while maintaining the fairness we can use some heuristic methods. Here we are using the following heuristic algorithm for eliminating edges.

Algorithm

Input: A Bipartite graph connecting k connected components c1, c2... ck. Each component c_i has α_i kid nods and β_i gift nodes on left and right side respectively with at most $2\alpha_i + \beta_i$ edges.

Output: A structure in which each c_i has at most $\alpha_i + \beta_i$.

1: For all connected components in the bipartite graph do steps 2 to 9

2: Sort all the kid nodes of the component in the non increasing order of the degree value

3: Set the counter value of all the nodes as Zero.

4: **while** Number of edges $\geq \beta + \alpha$ do

5: Select the kid node with highest degree value and lowest counter value

6: Check whether all the incident edges are coming from pendant vertices from gift side , then increase the counter by one and continue .

7: Select edge with minimum fractional value coming from a non pendant vertex and remove it.

8: Decrease the value of degree by one and increase the value of counter by one

9: **end while**

Here we get a bipartite graph with exactly m + n edges. This heuristic method will reduce the fractional optimal value FOPT of each kid by a factor γ. That is the new value of T becomes FOPT − γ that can be represented as T^{i}.

INTEGER ROUNDING OF FRACTIONAL SOLUTION

The integer rounding of fractional optimal solution can be done by different methods. Here in this report we suggest some rounding methods for our capacitated Santa Claus problem. The rounding method we used here is credited from Bezakova and Dani [2005]. Bezakova and Dani show that, if FOPT is fractional optimum and y is the integer solution then mini v_i (yi) ≥ max (0, FOPT − $\max_{ij} v_i$ (j)), where y_i denotes items allocated to player i in integer solution, v_i (yi) denotes the additive valuation of set y_i for player i and v_i (j) denotes the value of item j for player i. Here in this case we had been removed some edges and thus our FOPT becomes FOPT −γ and here after we can represent it as $FOPT^{|}$. Therefore our fractional optimum becomes $FOPT^{|}$ and after applying the same rounding method specified in the Bezakova and Dani[2005] the integer solution y becomes $\min_i v_i(y_i) \geq \max(0, FOPT^{|} - \max_{i,j} v_i(j))$

Lemma: *If G is a bipartite pseudo tree with A and B as the vertex sets and E as the edge set, then there is a set S such that ∀j ∈A, S covers j by exactly one edge and ∀i ∈B, S covers i by at least deg(i) − 1 edges*

Proof: The following function calculate (G, S) is used to find such a set S. Initially S is empty.

Function calculate (G, S)

If (∃ a vertex, v ∈ A s.t, deg (v) = 1)

1: Let e be the edge with vertex v

2: Remove e and v from G and call the new graph G'

3: calculate (G', S ∪ e)

Else if (∃ a vertex, v ∈ B s.t, deg (v) = 1)

1: Let e be the edge with vertex v

2: Remove e and v from G and call the new graph G'

3: calculate (G', S)

Else

1: So G is not a tree, hence it must have one extra edge and thus must have a cycle. As G is bipartite, cycle must be of even length.

2: S=either of alternating edges in cycle

After creating the bipartite graph apply the algorithm. This algorithm ensures that while constructing set S we remove at most one edge incident to every player in set B and thus we lose at most $\max_j (v_i (j))$.

So, $\min_i v_i (y_i) \geq FOPT' - \max_{ij} v_i (j)$.

CONCLUSION AND OPEN DIRECTIONS

On the theoretical front, we studied the spectrum of Santa Claus problem with Uniform Case on one end and the Restricted Assignment Case on the other end. We were successful in analyzing this spectrum and we have implemented some algorithmic solutions. Based on the theoretical and practical background on Santa Claus problem, we proposed the capacitated Santa Claus problem and suggested some methods for solving it. The spectrum is still large and there are many problems inside the spectrum that needs to be analyzed. Some of those problems are the cases where every item can be liked by every player except m number of players. All the problems inside the spectrum can be studied by future researchers which might lead to better approximation guarantee on the Santa Claus problem. We used linear programming based approximation methods to solve the problem. Based on the integer rounding technique, the approximation guarantee may change. In capacitated Santa Claus, can we use randomized rounding to give a better-than linear approximation guarantee? So our work has opened up this spectrum and has given a starting direction in the analysis of this spectrum

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